Forests of Motzkin Trees

A Motzkin tree is either empty, has one child tree of the root, or has two child trees of the root. We can model these three cases by structuring a finite set: either the set is empty, or we mark a point and put a Motzkin tree on the rest, or we mark a point, split the rest in two and put at Motzkin tree on each part.

$$M(z) = 1 + zM(z) + zM(z)^2$$

Solving for $M(z)$ gives

$$M(z) = \frac{1}{2z}\left(1 - z \pm \sqrt{(z-1)^2 - 4z}\right)$$

and we choose the negative square root to make the Taylor coefficients positive. So we have the sum

$$M(1) = \sum M_k = 1 + 2 + 4 + 9 + 21 + 51 + \cdots = -i$$

and it is conceivable that there is an isomorphism $M^5 \cong M$ between 5-tuples of Motzkin trees and single Motzkin trees. In fact, this isomorphism exists and can be explicitly constructed. In this derivation, $=_1$ means we apply the function $M \xrightarrow{f} 1 \cup M \cup M^2$ which takes a tree and returns $\{\}$ if the tree is empty, the subtree of the root if the root has only one child, and the ordered pair of subtrees if the root has two children. $=_2$ denotes an application of this function's inverse.

(isomorphism is on the next page, don't read it if you don't want to spoil the puzzle!)

$$\begin{aligned}
M &=_1 1 + M + M^2 \\
&=_1 1 + 2M + M^2 + M^3 \\
&=_2 2M + M^3 \\
&=_1 2M + M^2 + M^3 + M^4 \\
&=_2 M + M^2 + M^4 \\
&=_1 M + M^2 + M^3 + M^4 + M^5 \\
&=_2 M^2 + M^4 + M^5 \\
&=_1 M^2 + M^3 + M^4 + 2M^5 \\
&=_2 M^3 + 2M^5 \\
&=_1 M^3 + M^4 + 2M^5 + M^6 \\
&=_2 M^4 + M^5 + M^6 \\
&=_2 M^5
\end{aligned}$$